Two-dimensional system of charges in cylindrical traps

Hiroo Totsuji

Department of Electrical and Electronic Engineering, Okayama University, Tsushimanaka 3-1-1, Okayama 700, Japan

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Some phenomena induced by placing an electrode at the center of a cylindrical apparatus confining ions or electrons, such as magnetic and electrodynamic traps, are analyzed on the basis of an electrostatic model. It is shown that a purely two-dimensional system of charges can be formed on a cylindrical surface under appropriate conditions.

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There have been observed various structures of the nonneutral systems of charged particles at low temperatures in magnetic and electrodynamic traps [1, 2]. The structure changes with the system parameters such as the number of charges. The purpose of this paper is to analyze some phenomena related to the introduction of an electrode at the center of these traps, especially the formation of two-dimensional plasmas, its electrostatic potential being regarded as a new parameter. The results may apply to both magnetic traps (Penning and electron traps) and electrodynamic traps (Paul traps).

We assume that the trap has the cylindrical symmetry at least in the central part of the system. Our proposal is to apply a voltage ϕ_0 (relative to outer conductors) to an electrode placed at the symmetry axis as a source of an electrostatic potential

$$U_{\rm ext}(\{R_i\}) = -q\phi_0 \sum_i \ln\left(\frac{R_i}{L}\right). \tag{1}$$

Here q is the charge, R_i is the radius of the particle i at $\mathbf{r}_i$ in the cylindrical coordinates $\mathbf{r}_i = (R_i, \theta_i, z_i)$, and the zero of the potential is taken at the outer conductor of radius L.

In the magnetic traps, the Hamiltonian of our system is written as $H = H_0 + U_{\rm ext}(\{R_i\})$ where $H_0 = (m/2)\sum_i \mathbf{v}_i^2 + U_{\rm int}(\{\mathbf{r}_i\})$, $U_{\rm int}(\{\mathbf{r}_i\})$ being the mutual Coulomb interaction. The statistical distribution in our system is described by the Hamiltonian in the rotating system [3]

$$\begin{aligned} &H_0 + U_{\rm ext}(\{R_i\}) - \omega M_z \\ &\quad = E_{\rm kin} + U_{\rm int}(\{\mathbf{r}_i\}) + U_{\rm ext}(\{R_i\}) \\ &\qquad -\frac{m\omega}{2}(\Omega-\omega)\sum_i R_i^2, \end{aligned} \tag{2}$$

where $\Omega = qB/mc$ is the cyclotron frequency; ω, the angular velocity of solid rotation; $E_{\rm kin}$, the kinetic energy in the corotating frame; and M_z, the parallel component of the total canonical angular momentum. Note that ω is negative (positive) when $q > 0$ ($q < 0$). We have a system of charged particles confined by the effective background charges of the uniform density $m\omega(\Omega-\omega)/2\pi q^2$. We denote the effective confining force constant $-m\omega(\Omega-\omega)$ (> 0) by k.

In the electrodynamic traps, the force constant k is related to the oscillating electric field [4]. The Hamiltonian, however, takes the same form as (2) and we may apply our results on the structure equally well to these traps.

The structure of trapped charges in our system at sufficiently low temperatures may be analyzed by extending the model developed by Barrat and the author [5,6] to include $U_{\rm ext}$. In the case of $\phi_0 = 0$, this model has been successful in reproducing the structures in cylindrical traps. The essential ingredient of the model is to properly account for the effects of discreteness (correlation) in both radial and azimuthal directions. The former is taken into account by assuming the shell structure, and the latter, by considering the correlation energy of the charges on the cylindrical surface.

We may thus write the potential energy per particle in the form

$$\frac{1}{2}k\sum_{i=1}^{N}\left(\frac{n_i}{n}\right)R_i^2 - q\phi_0\sum_{i}^{N}\left(\frac{n_i}{n}\right)\ln\left(\frac{R_i}{L}\right) - \frac{q^2}{n}\left[2\sum_{i>j}^{N} n_i n_j \ln\left(\frac{R_i}{L}\right) + \sum_{i=1}^{N} n_i^2 \ln\left(\frac{R_i}{L}\right) - \sum_{i=1}^{N} n_i^2 c (R_i n_i)^{-\frac{1}{2}}\right]. \tag{3}$$

Here N is the number of shells, R_i and n_i are the radius and number of charges (per unit length) of the shell i, and $n = \sum_i n_i$. The last term in square brackets is the correlation energy within the shell approximately expressed by that of planar lattice with $c = -0.782\,13$ [7].

Noting that the effect of ϕ_0 is equivalent to the existence of charges at the center, we define the effective charge density $n_{\rm eff}$ per unit length by

$$n_{\rm eff} = n + \phi_0/2q. \tag{4}$$

The potential energy per particle in our model is thus rewritten as

$$q^2\frac{n_{\rm eff}^2}{n}\sum_{i=1}^{N}[\tfrac{1}{2}k'(f_i-f_{i-1})R_i'^2-\phi_0'(f_i-f_{i-1})\ln R_i'-(f_i^2-f_{i-1}^2)\ln R_i'+c(f_i-f_{i-1})^{\frac{3}{2}}R_i'^{-\frac{1}{2}}]+q^2n\left(1+\frac{\phi_0}{qn}\right)\ln L. \tag{5}$$

Here $k'=k/q^2n_{\rm eff}^3$, $\phi_0'=\phi_0/qn_{\rm eff}$, $R_i'=n_{eff}R_i$, and $f_i=\sum_{j(\le i)}n_j/n_{\rm eff}$. By replacing f_i by $f_i'=f_i+\phi_0'/2$, we can formally eliminate the term proportional to ϕ_0' and rewrite the essential part of (5) into the same form as in the case of $\phi_0'=0$; the only difference is that $f_0'=\phi_0'/2$ instead of $f_0=0$. The number of shells and the position and population of each shell are calculated by minimizing (5) with respect to N, $\{R_i'\}$, and $\{n_i/n_{\rm eff}\}$.

Our system is characterized by two parameters, k' and ϕ_0'. Considering an imaginary process of gradually collecting charges on inner shells to the center, starting from the state of $\phi_0'=0$, we expect that the comparison of systems for finite values of ϕ_0' with those for $\phi_0'=0$ at the same value of k' may be useful.

In Fig. 1 we observe that the positions of shells are approximately given by those in the case of $\phi_0'=0$ [6] especially for outer shells. Given the number of shells, the diagram of radius for $\phi_0'\neq 0$ may be formed by taking the necessary number of branches for $\phi_0'=0$ beginning with the outermost shell. In Fig. 2 we see that the deviations in the population are somewhat larger than those in positions.

The number of shells decreases with the increase of the external potential ϕ_0'. The boundaries of these transitions are shown in Fig. 3 by broken lines as functions of parameters k' and $\phi_0'/2$. Note that $\phi_0'/2$ is the ratio of charges on the electrode to total effective charge.

In Fig. 3, we have also plotted the values of $f_{M-N}=\sum_{j(\le M-N)}n_j/n$ in the case of M shells for $\phi_0'=0$ [6] as functions of k': f_{M-N} is the fraction of charges on inner $M-N$ shells. Observed at radius R, the effect of uniformly charged shells with radii smaller than R is equivalent to the charges of the same amount placed at the center. Therefore we may gather the charges of inner $M-N$ shells to the center with little change in the structure of outer N shells. As a result, we may have a system which is characterized by the parameters k' and $\phi_0'/2=f_{M-N}$. We thus expect that, on the line f_{M-N}, we have N shells in our system with an electrode. In the domain between the lines f_{M-N} and f_{M-N-1}, however, we have two possibilities; either to maintain still $N+1$ shells with reduced population on the innermost shell or to change to the state of N shells, charges on the vanishing shell moving to the next.

For finite values of ϕ_0', we observe that the boundary between $N+1$ shells and N shells lies about halfway between the lines f_{M-N} and f_{M-N-1}. When about half of the charges on the innermost shell are absorbed to the central electrode, the shell disappears and the rest of the charges there join the next (and now innermost) shell.

Here we note that when the potential is sufficiently large we have a state of one shell. In the case of a very strong potential, the charges may sit at the minimum of the sum of the confining force and the potential of the electrode. We thus have a single layer or a purely two-dimensional system of charges on a cylindrical surface. This is in contrast to the charges in the shells realized in the cylindrical or spherical confinements. In the latter case, the singleness of the layer is not guaranteed in principle.

When we increase the areal number density of charges on the minimum surface, the charges will cease to be single layered at some critical value. They increase average mutual distance and lower mutual Coulomb energy with the expense of the confining and external potentials.

In the case of cylindrical confinement without an electrode, the charges are lined up at the center when the

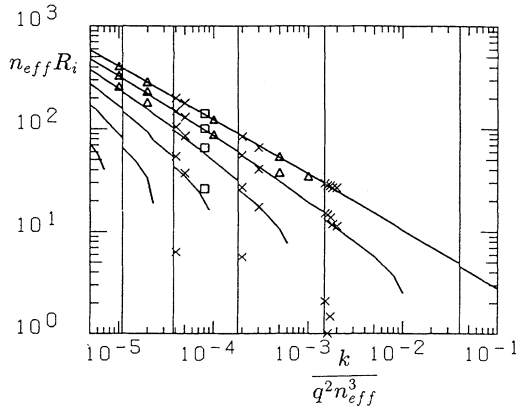

FIG. 1. Positions of shells. Solid lines are values $\phi_0'=0$ [5, 6]. Triangles are those with $\phi_0'/2=0.25$. Squares [8] and crosses [6] are results of numerical experiments for $\phi_0'=0$.

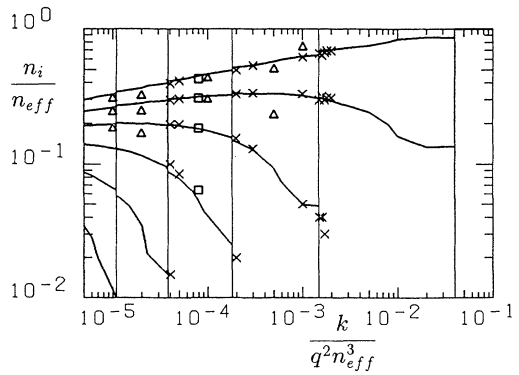

FIG. 2. Populations of shells. Symbols are the same as in Fig. 1

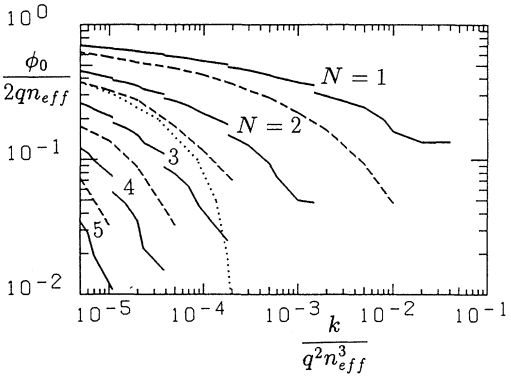

FIG. 3. Number of shells (N) changes at broken lines. Values of f_{M-N} ($M = 1, 2, \ldots, 6$) for $\phi_0' = 0$ are plotted and compared with the domain where the number of shells is equal to N. A limit of stability of the two-dimensional system is shown by the dotted line.

effective force constant is sufficiently large [8, 5, 9, 10]. With the increase of the number of charges (per unit length), they form successively a twofold helix (zigzag), a threefold helix, and so on. The breakdown of a single layer in our system corresponds to the transition from the line to the twofold helix.

When the radius is large, the instability of the 'transverse' (or normal to the layer) phonon may give a limit of areal density for single layer. For large radii, the dispersion relation of the phonon is given by

$$-\frac{m}{u}\frac{d^2u}{dt^2} = 2k + 2\sum_{\mathbf{P}_i \neq \mathbf{0}} \frac{q^2}{P_i^3}[\cos(\mathbf{K}\cdot\mathbf{P}_i) - 1], \qquad (6)$$

where $\mathbf{P}_i$ and $\mathbf{K}$ are the two-dimensional lattice and the wave vectors, respectively. In the triangular lattice, the second term gives the minimum $-13.35q^2/a^3$ (a being the lattice constant) for the wave vector $\mathbf{K}_0$ at the corner of the first Brillouin zone as shown in Fig. 4. The displacement u corresponds to the separation of the triangular lattice into three larger triangular lattices.

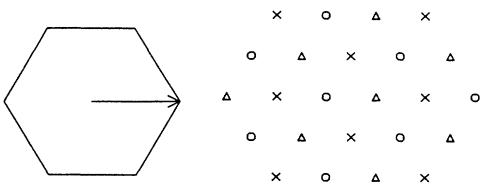

FIG. 4. Wave vector of the most unstable mode in the Brillouin zone and corresponding deformation; triangles and crosses are equally displaced from and to the plane, respectively.

In our parameters, the criterion for stability to this deformation is written as

$$k'(1 + 0.5\phi_0')^3/(1 - 0.5\phi_0')^6 > 2.127 \times 10^{-4}. \qquad (7)$$

This condition is plotted in Fig. 3 by a dotted line and we see that it is always satisfied by the state of one shell. We thus have a possibility to realize two-dimensional systems of charges by applying a potential to the central electrode in cylindrical traps.

It should be noted, however, that (7) gives a limit of stability: There may exist another mechanism which destroys the single layer. We also note that in the case of purely one-dimensional confinement where confining force is proportional to $kx^2/2$, the coefficient 2 before k in (6), coming from the balance between $kR^2/2$ and $-q\phi_0 \ln(R/L)$, is to be replaced by 1.

The systems of charges discussed here are expected to be classical except for very unusual values of the parameters. They provide us with a possibility to directly observe the formation of classical two-dimensional lattices of charges.

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